Continuum percolation threshold for permeable aligned cylinders and opportunistic networking

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***Abstract*—We consider the critical percolation threshold for aligned cylinders, which provides a lower bound for the required node degree for the permanence of information in opportunistic networking. The height of a cylinder corresponds to the time a node is active in its current location. By means of Monte Carlo simulations, we obtain an accurate numerical estimate for the critical reduced number density, $\eta_c \approx 0.3312(1)$ for constant height cylinders. This threshold is the same for all ratios of the height to the diameter of the base, and corresponds to the mean node degree of 1.3248 in opportunistic networking, which is clearly below the percolation threshold of 4.51 above which a gigantic connected component emerges in the network.**

I. Introduction

In the continuum percolation problem, objects of given shape are placed according to a k-dimensional Poisson process. Two objects are neighbors if they overlap. The percolation is said to occur if with a positive probability a random object belongs to an infinite cluster, which exists only when the density of objects is above the so-called critical percolation threshold. Among many other fields, continuum percolation theory has been applied to study wireless networks. A wealth of results exist for two-dimensional percolation since the early work of Broadbent and Hammersley [1]. In the basic continuum model, the objects are permeable discs, for which the first estimate of the critical percolation threshold was given by Gilbert in [2] (in the context of wireless multi-hop networks). Since then, the computing power has increased enormously, as have the techniques to determine the quantity of interest. Consequently, a very accurate estimate for the threshold has been provided by Quintanilla et al. [3], $\phi_c = 0.6763475(6)$.

In three dimensions, the basic case is the one with permeable (interpenetrating) spheres with a fixed radius, for which an accurate estimate was obtained by Lorenz and Ziff [4], $\phi_c = 0.289573 \pm 0.000002$. Similarly, Baker et al. considered both aligned and isotropically oriented squares and cubes [5]. The aligned cubes are of interest to us, for which the criticality threshold is $\phi_c = 0.2773 \pm 0.0002$. Balberg and Binenbaum [6] considered a system which resembles closely our model. In particular, they consider three-dimensional permeable “sticks” comprising a cylinder body which is capped by two hemispheres of equal radius. The difference is the additional caps and the alignment of the sticks; in our case the cylinders will be strictly aligned.

The critical density for percolation has been expressed by different quantities in the literature. Above we used the *volume fraction* ϕ, defined as the fraction of the space the objects occupy. The *reduced number density* is a dimensionless quantity defined as the product,

$$\eta = n \cdot V,$$

where n and V are the number density and volume of the objects. The elementary relationship between these is

$$\phi = 1 - e^{-\eta}.$$

The critical values of η and ϕ are denoted η_c and ϕ_c.

In this paper, we study the continuum percolation of permeable aligned cylinders, which is related to the permanence of information in opportunistic networking schemes. In Section II, we first derive bounds for the critical reduced number density η_c based on the known results for spheres and cubes, and then, by means of Monte Carlo simulations, also give an accurate estimate for η_c and the corresponding mean node degree ν in opportunistic networks. The percolation threshold gives a lower bound for the permanence of information, which is then, in Section III, studied directly by simulating an actual content dissemination scheme. Section IV concludes the paper.

A. Background

In the floating content application and related concepts, information is stored in mobile wireless nodes which duplicate it whenever they meet other nodes within the area of interest [7]. As nodes eventually depart the area, the information is bound to be lost when the last information carrying node departs before passing the information further.

An abstract model for the availability of the information is as follows. Consider an *ad hoc* network comprising nodes which appear in the plane according to a Poisson process and remain stationary for a random time before they disappear (depart). In other words, we assume that communication takes place only when two nodes are stationary and within each other’s transmission range. For the latter, we assume Gilbert’s disc model (boolean model), where communication is possible whenever the distance between two nodes is at most the transmission range d.

The described system constitutes a stochastic process where discs corresponding to nodes appear in the (x, y)-plane. The radius of the discs is related to the transmission range d. For a succesful communication, the distance between two nodes must be at most d. The equivalent *radius for percolation* is $r = d/2$, i.e., the transmission range d corresponds to the diameter $2r$ with regards to percolation.

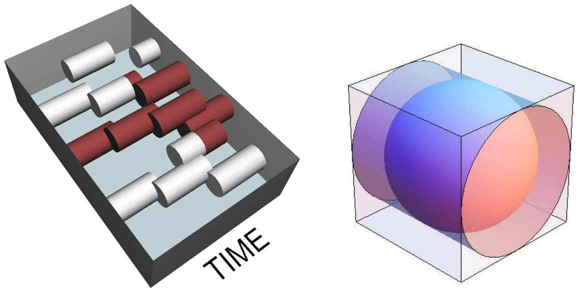

Fig. 1. Left: Aligned cylinders and dissemination of the information (red cylinders) in time. The three cylinders at the bottom right are connected to the red component, but the information flows only in the direction of positive time-axis. Right: Derivation of the lower and upper bound for the percolation threshold for aligned cylinders.

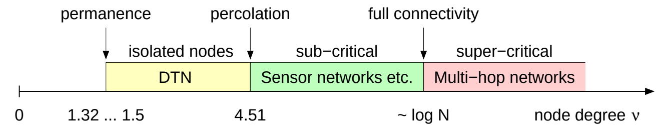

Fig. 2. Regimes for different networking concepts. Below the percolation bound one has to resort to delay tolerant networking (DTN), while below the permanence bound no meaningful communication is possible.

Interpreting the time as z-axis yields sample realizations as depicted in Fig. 1. The dark red color in a cylinder indicates that the corresponding node has the information at the given time instant. The key observation is that *the phenomenon for the availability of information at time t (or forever) corresponds to a directed percolation along the z-axis* for aligned cylinders. Hence, there are two important differences to the classical percolation problem: (i) objects are aligned instead of being isotropically oriented, and (ii) percolation is directed due to the causality along the time-axis. In the next section, we relax the latter and study the classical percolation problem with aligned objects. The obtained results serve as strict lower bounds for the critical node density with respect to the permanence of the information.

II. Analysis of Percolation Bound

In this section, we assume that the stopping time T of a node is constant and neglect the causality. Let h be the corresponding length in the z-direction in the three-dimensional spatial model. We note that one can choose the scale for z-axis arbitrarily and the resulting process is a Poisson point process in three dimensions.

Let λ denote the arrival rate per unit area, so that $n_2 = \lambda \cdot T$ corresponds to the number density of discs in the (x, y)-plane at any given point of time. The mean node degree (number of neighbours) is

$$\nu = n_2 \, \pi \, d^2.$$

The number density of objects in three dimensions depends on the choice of scale and reads

$$n = \lambda \cdot T/h = n_2/h.$$

Combining the above gives

$$\nu = n_2 \pi d^2 = (n_2/h)(\pi h d^2) = 4nV = 4\eta,$$

where V is the volume of the cylinder. Thus, the elementary relationship between ν and the reduced number density of cylinders η in three dimensions is

$$\nu = 4\,\eta,$$

and the problem of the critical mean node degree reduces to that of finding the critical percolation density for cylinders.

A. Lower and Upper Bounds

Let η_a denote the unknown critical parameter of some shape a, and η_b the known critical parameter for some other shape b. In general, it holds that

$$(\check{V}_a/V_b)\,\eta_b \le \eta_a \le (\hat{V}_a/V_b)\,\eta_b,$$

where $\hat{V}_a$ is the volume of such an object of type a that wholly contains an object of type b with volume V_b, and similarly, $\check{V}_a$ denotes the volume of such an object of type a which wholly fits inside an object of type b. The tightest bound is obtained when the two volumes are as close to each other as possible.

In the present case, we derive lower and upper bounds for the critical reduced number density for aligned cylinders based on the known results for spheres, $\eta_{\text{sphere}} = 0.341889(3)$ [4], and aligned cubes, $\eta_{\text{cube}} = 0.3248(3)$ [5]. The critical reduced number density for aligned cylinders is insensitive to the ratio between the height of the cylinder and the diameter of the base, which allows an optimization over the mentioned ratio. The tighest upper bound for the aligned cylinders is obtained by taking the object b to be a sphere, while the lower bound is obtained with a cube, as illustrated in Fig. 1. In terms of the mean node degree,

$$1.02 < \nu_c < 2.052. \tag{1}$$

Unfortunately, these bounds are rather loose, which is explained by the fact that the volume of the smallest cylinder containing a sphere is 1.5 times the volume of the latter and that the volume of the largest cylinder contained in a cube is about 0.79 times the volume of the latter.

However, as cylinder can be thought as a shape between a sphere and a cube (most obvious if the cylinder's height equals the diameter of the base, a choice that, as said, does not affect η_{cyl}), one can expect that η_{cyl} is somewhere between the corresponding values of sphere and cube[1], $0.3248 \le \eta_{\text{cyl}} \le 0.3419$, or in terms of the mean node degree,

$$1.2992 \le \nu_c \le 1.3676. \tag{2}$$

The average node degree ν for connectivity in the plane is about 4.51 neighbours. This means that the *permanence of information* is achieved with a clearly lower node density than what an instantaneous end-to-end communication requires.

Fig. 2 illustrates the regimes of different wireless networking concepts as a function of (mean) node degree. The three regions can be identified in general, while the shown numerical values correspond to the stationary model where communication takes place only when a node is not moving.

[1] Incidentally, $\eta_{\text{sphere}} + \eta_{\text{cube}} \approx 2/3$.

TABLE I
PARAMETERS FOR MONTE CARLO SIMULATION.

shape	dimensions	volume	reduced number density η_c	volume fraction ϕ_c	
cube	$\ell = 1$	1	0.3248 ± 0.0003	0.2773 ± 0.0002	[5]
sphere	$r = 1/2$	$\pi/6$	0.341889 ± 0.000003	0.6763475 ± 0.0000006	[4]
cylinder	$r = 1/2$, $h = 1$	$\pi/4$	0.3312 ± 0.0001	0.2819 ± 0.0002	(Sect. II-B)

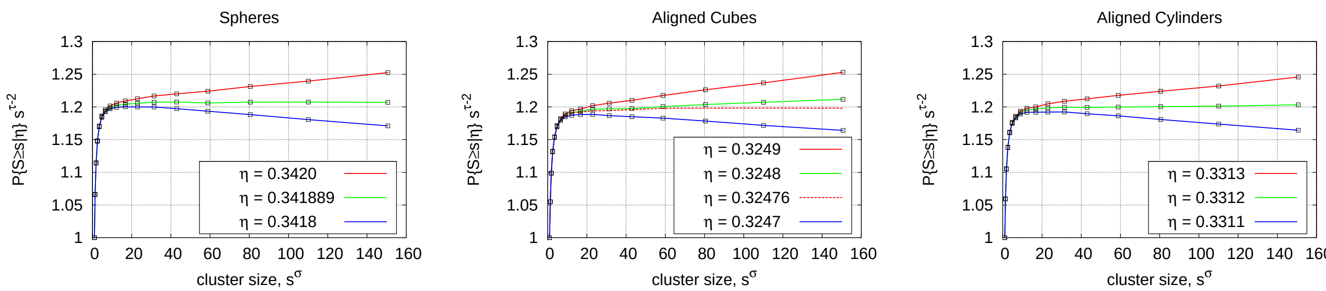

Fig. 3. Monte Carlo simulation results for percolation with spheres, aligned cubes and aligned cylinders.

B. Monte Carlo Simulations

Next we carry out Monte Carlo simulations based upon the method by Baker et al. [5], which again was based on the work by Leath [8] and Lorenz and Ziff [4]. First we briefly describe the methodology, while for more details we refer to the aforementioned work.

1) Methodology: Let random variable S denote the size of the cluster a randomly chosen object belongs to. The idea is to determine the cluster size distribution by simulations and then utilize its known asymptotic behavior,

$$\mathrm{P}\{S \geq s \mid \eta\} \sim A s^{2-\tau} f((\eta - \eta_c) s^{\sigma}),$$

where τ and σ denote universal exponents and A is some (non-universal) constant. In three dimensions [9],

$$\tau = 2.18906 \pm 0.00006,$$
$$\sigma = 0.4522 \pm 0.0008.$$

Near the percolation threshold, where $\eta \approx \eta_c$, Taylor series for $f(x)$ is $f(x) = 1 + Bx + \ldots$, which gives

$$\mathrm{P}\{S \geq s \mid \eta\} \cdot s^{\tau-2} \sim A + AB(\eta - \eta_c)s^{\sigma} + \ldots,$$

i.e., the quantity on the left-hand side becomes a constant when $\eta = \eta_c$, which can be used to determine η_c.

In [5] Monte Carlo simulations are carried out in $L \times L \times L$ box, which is subdivided to L^3 unit boxes. This enables an efficient procedure where (i) search of the neighbors is limited to the surrounding unit boxes and (ii) the unit boxes are populated only on demand according to the Poisson point process. For odd L, a center box is well-defined and a test object can be placed initially in the middle of it. Samples of the cluster size S are binned so that the kth bin corresponds to the number of samples with cluster size $s \geq 2^k$, $k = 0, 1, \ldots$. As samples reaching the boundary would cause a bias to the results, only bins up to the smallest bin having such samples are included in the results [5]. The shortcoming is that one ends up using relatively high values for size L while most of the unit boxes remain unused.

In this work, we consider the aligned shapes given in Table I. As we limit ourselves to study objects that fit inside a unit box, a given object can overlap only with objects located within the $3 \times 3 \times 3$ box around it. For each new object added to the cluster, we first ensure that the 26 surrounding boxes are populated, and then add the connected objects to the cluster. This is repeated recursively until no new objects are included. Our implementation, written in C, uses breadth-first search to this end. The pseudo random numbers are generated using Mersenne Twister [10].

A fixed $L \times L \times L$ box would enable a fast access to unit boxes, e.g., through an appropriately chosen array. However, we do not use such, but instead restrict the search only by defining a maximum cluster size $s_{\max}$ corresponding to the highest bin we are interested in; for which a rational choice is $s_{\max} = 2^K$ for some K. Unit boxes are stored in a hash of lists, which allows a cluster to grow dynamically to all directions. These modifications improve the mean running time significantly allowing larger sample sizes and higher numbers of bins. At the same time, it also relieves us from adjusting the box size parameter L *a priori*. For example, the side length of $L = 161$, used in [5], corresponds to about 4.2 million unit boxes, while a comparable $s_{\max}$ achieved was only $1024, \ldots, 4096$ nodes.

2) Numerical results: We choose $K = 16$ so that the last bin corresponds to cluster sizes of 65536 nodes and higher. For each shape and reduced number density η, we generate $N = 10^6$ clusters to obtain an empirical ccdf of the cluster size S. Fig. 3 illustrates the results. From the literature we know the critical reduced number density η_c for spheres and aligned cubes (see Table I), and thus the corresponding curves serve as a validation of the simulation framework. In passing, we note that according to our experiments $\eta_{\text{cube}} = 0.32476(4)$, which gives one additional digit to the result given in [5].

To the best of our knowledge, the critical density for percolation with aligned cylinders has not been determined before. According to our Monte Carlo simulations,

$$\eta_{\text{cyl}} = 0.3312(1),$$

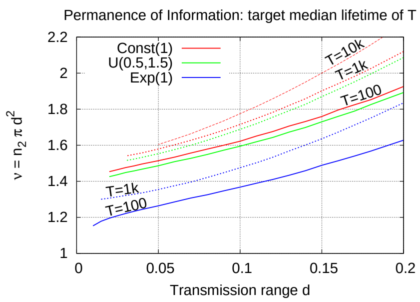

Fig. 4. Numerical results for a finite target median lifetime in unit disc with Gilbert's disc model and different stopping time distributions.

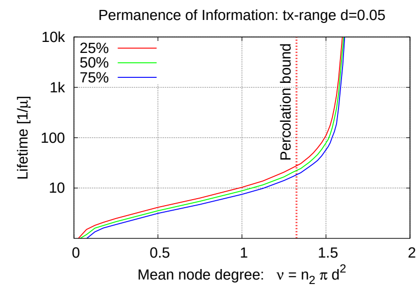

Fig. 5. Information lifetime as a function of the mean node degree ν. The y-axis is on logarithmic scale and the percolation bound refers to (3).

which corresponds to the mean node degree of

$$\nu_c = 1.3248(4). \tag{3}$$

This value also fits nicely within the conjectured bounds (2).

III. Simulation of the System

The continuum percolation model considered in the previous section provided a lower bound for the critical mean node degree for the permanence of information. In this section, we simulate the actual stochastic process of information carrying nodes in the plane. At time $t = 0$, all nodes present in the area (unit disc) are given the information and we record how long the information remains available in the given area. Other simulation parameters are the arrival rate λ, the stopping time distribution, and transmission range d. When d tends to zero, the model approaches that of the percolation model.

The curves in Fig. 4 correspond to parameter values yielding a median lifetime of $T = 100$, 1k, 10k. The mean stopping time (lifetime of a node) is normalized to 1. The constant stopping time corresponds to the results of the previous section. We observe that the critical mean node degree ν_c appears to be about $1.3 \ldots 1.5$ depending on the stopping time distribution, which matches well with the lower bound (3). Moreover, varying stopping times also clearly improves the lifetime of the information, as expected.

In Fig. 5, where the transmission range is fixed, $d = 0.05$, and the stopping time is constant, x-axis represents the mean node degree and on y-axis is the lifetime on logarithmic scale. The three curves correspond to the 25%, 50% and 75% quartiles. We observe that lifetime increases gradually below the percolation bound (3) until, at about 1.5, it suddenly skyrockets. This phenomenon corresponds to crossing the permanence bound.

IV. Conclusions

Percolation theory has been applied to numerous fields and problems. In the context of wireless networking, a two-dimensional percolation model has been traditionally used to analyze the connectivity and capacity in multi-hop networks [2], [11], [12]. In this paper, we have demonstrated that the three-dimensional percolation model can be utilized to study the permanence of information in opportunistic networking schemes. The third dimension corresponds to the time in this case. Assuming Gilbert's disc model for transmission range, and constant node lifetimes (stopping times), the resulting percolation model comprises identical aligned cylinders.

Consequently, we then first derived bounds for the critical reduced number density of aligned cylinders, and subsequently also gave an accurate numerical estimate for it, $\eta_{\text{cyl}} = 0.3312$, corresponding to the mean node degree of $\nu_c = 1.3248$. The result matches well with the simulation results of the actual opportunistic networking scheme, where nodes collectively store and disseminate information content in best effort fashion.

Acknowledgements

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